

About the Malmquist bias in the determination of H_0 and of distances of galaxies

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Abstract. We provide the mathematical framework which elucidates the way of using a Tully-Fisher (TF) like relation in the determination of the Hubble constant H_0 , as well as for distances of galaxies. The first step toward the comprehension of this problem is to define a statistical model which accounts for the (linear) correlation between the absolute magnitude M and the line width distance estimator p of galaxies, as it is observed. Herein, we assume that $M = a.p + b - \zeta$, where ζ is a random variable of zero mean describing an intrinsic scatter, regardless of measurement errors. The second step is to understand that the calibration of this law is not unique, since it depends on the statistical model used for describing the distribution of variables (involved in the calculations). With this in mind, the methods related to the so-called Direct and Inverse TF Relations (herein DTF and ITF) are interpreted as maximum likelihood statistics. We show that, as long as the same model is used for the calibration of the TF relation and for the determination of H_0 , we obtain a coherent Hubble's constant. In other words, the H_0 estimates are not model dependent, while the TF relation coefficients are. The choice of the model is motivated by reasons of robustness of statistics, it depends on selection effects in observation which are present in the sample. For example, if p -selection effects are absent then it is more convenient to use a (newly defined) robust statistic, herein denoted by ITF*. This statistic does not require hypotheses on the luminosity distribution function and on the spatial distribution of sources, and it is still valid when the sample is not complete. Similarly, the general above results apply also to distance estimates of galaxies. The difference on the distance estimates when using either the ITF or the DTF model is only due to random fluctuations. It is interesting to point out that the DTF estimate does not depend on the luminosity distribution of sources. Both statistics show a correction for a bias, inadequately believed to be of Malmquist type. The repercussion of measurement errors, and additional selection effects are also analyzed.

Key words: galaxies : distances and redshifts – methods : statistical

1. Introduction

Herein, we regard the Tully-Fisher (TF) relation for spiral galaxies in optical, Tully & Fisher (1977), and the Faber-Jackson relation for E galaxies, Faber & Jackson (1976), as a single law providing us with an estimate of the absolute magnitude $M = ap + b$, where p is called *line width distance estimator*. The determination of the Hubble constant H_0 , when a line width distance estimator is involved has long been discussed with respect to the Malmquist bias by different authors without reaching yet a general agreement (see e.g., Bottinelli et al. 1986a,1986b,1988a,1988b; Giraud 1985,1987; Gouguenheim et al. 1989; Jacoby et al. 1988; Lynden-Bell et al. 1988; Pierce & Tully 1988; Tammann 1987,1988; Sandage 1988a,1988b; Teerikorpi 1975,1982,1984,1987,1990; Tully 1988). The aim of this work is to enlighten on this problem through a theoretical point of view, and to provide us with rigorous formulas for ongoing applications. In this first paper, we seek a mathematical framework which gives fair rudiments for discussing on the use of the *Direct* Tully-Fisher (DTF) relation and the *Inverse* Tully-Fisher (ITF) relation, which both interpret as a choice of a fitting technique.

According to Bigot & Triay (1990b), one must keep in mind that a technique of fitting is intimately related to a statistical model. Namely, the related statistics (or estimates) are warranted as long as the values distribution of variables involved in the calculation is correctly described by such a model. Hence, we understand that a statistical model is as a matter of fact required for arguing on the use of either the DTF relation or the ITF relation. However, it must be noted that if a statistical model is available then nothing prevents us to use solely the *maximum likelihood* (ML) technique. Such an approach has the advantage of providing us unambiguously with a unique fitting technique. Moreover, the related statistics give unbiased estimates of model parameters, as long as the statistical model takes into account the selection effects.

Therefore, according to above precepts, in Section 2, we define the statistical model which describes the distribution

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of variables involved in the determination of H_0 . In Section 3, we derive the statistics used for the calibration of the M - p relation and the estimation of H_0 . The influence of selection effects on these estimates is also analyzed. In Section 4, we investigate the repercussion of measurement errors. Section 5 enlightens on the definition of a reliable distance estimate of galaxies. It is strongly recommended to read the notations and useful formulas given in Appendix A, these features are addressed throughout the text by means of symbol “Def.”.

2. The basic model

Herein, we specify the probability density (pd) describing the distribution of variables involved in the calculation. These variables are related to intrinsic quantities of sources (galaxies), which are :

- the absolute magnitude M ,
- the line width distance estimator p ,
- the distance r from the observer,
- and the radial velocity v (corrected for solar motion).

For reasons of simplicity in calculations, instead of using the variables r and v , it is more convenient to use the distance modulus

$$\mu = 25 + 5 \log r, \quad (1)$$

where r is given in Mpc, and a similarly defined quantity

$$\eta = 25 + 5 \log v. \quad (2)$$

If the peculiar velocities of sources are neglected then the Hubble law reads

$$\eta = \mu + \mathcal{H}, \quad (3)$$

where $\mathcal{H} = 5 \log H_0$. This equation shows that the variables μ and η define the same quantity¹. If no evolutionary effect of sources is present then the distribution of intrinsic quantities M and p are independent on μ (since a distance corresponds to a time-shift). Thus, regardless of capacities in observation, the theoretical pd describing the distribution of above variables can be written as follows

$$dP_{\text{th}} = F(M, p) dM dp \kappa(\mu) d\mu, \quad (4)$$

see (Def.1), where $\kappa(\mu)$ accounts for the distribution of galaxies in space and $F(M, p)$ for the M - p distribution, i.e., the TF diagram. The projection of the TF pd onto the M -axis, resp., the p -axis, provides us with the *luminosity distribution function*,

$$f_M(M; M_0, \sigma_M) = \int F(M, p) dp, \quad (5)$$

resp., a pd function (pdf) describing the distribution of p 's values,

$$f_p(p; p_0, \sigma_p) = \int F(M, p) dM, \quad (6)$$

¹This rough description suffices for the present scope, although this model is refined in Triay et al. (1993c).

see (Def.1).

It is obvious that this statistical model, as defined by Eq. (4), must be improved for taking into account observational and selection effects (e.g., the sampling rules). The difficulty in detecting faint galaxies involves necessarily the apparent magnitude

$$m = M + \mu. \quad (7)$$

If the selection effects depend solely on the apparent magnitude, then the data distribution is defined by the following pd

$$dP_{\text{obs}} = \frac{\phi_m(m)}{P_{\text{th}}(\phi_m)} dP_{\text{th}}, \quad (8)$$

where $\phi_m(m)$ is called *selection function*², and $P_{\text{th}}(\phi_m)$ is a normalization factor, see (Def.3).

2.1. Working hypotheses

In order to achieve the statistical model, as defined by Eq. (8), we must specify the functions $\phi_m(m)$, $\kappa(\mu)$ and $F(M, p)$. It turns out that some results can be obtained though without a full description of the model, which provides an interesting feature for related statistics : (*robustness*).

Since the present scope of our analysis is to enlighten on the problem of biases, we limit on formulating simple hypotheses but sufficiently complete. This has the advantage of avoiding cumbersome calculations but providing us with fair rudiments common to real situations. A more realistic model is given in Triay et al. (1993a). In the following, for clearness in understanding, we apply ourself to specify always the working hypotheses used for each step. In general, the standard working hypotheses assume :

- h_1) *A magnitude limited complete sample*. This property means that the selection effect limits to a cutoff at a given limiting magnitude, herein designated by m_{lim} . Thus, the selection function reads

$$\phi_m(m) = \theta(m_{\text{lim}} - m), \quad (9)$$

where θ is the Heaveside distribution function.

It is obvious that

$$m_{\text{lim}} \geq \max_{k=1, N} \{m_k\}, \quad (10)$$

and (in practice) a possible choice is to assume the equality.

- h_2) *A uniform spatial distribution of sources*. The pdf of the distance modulus μ , see Eq. (1), related to a uniform distribution in an Euclidian space is given by

$$\kappa(\mu) \propto \exp(\beta\mu), \quad \text{where } \beta = \frac{3 \ln 10}{5}. \quad (11)$$

Let us mention that, while we focus on a uniform spatial distribution of sources, the following calculations and results are still valid for power law distribution $\beta \neq \frac{3 \ln 10}{5}$. In order to specify the TF pdf $F(M, p)$, we must describe accurately the TF diagram, i.e., the relation $p \mapsto M = \tilde{M}(p)$. The observations show that the data are distributed about a straight line (which visualizes the TF relation), that we denote (Δ_{TF}) . Thus this line is defined by equation

²It is a positive function which works as a filter response ($0 \leq \phi_m(m) \leq 1$) with respect to the apparent magnitude.

$$\tilde{M}(p) = a.p + b. \quad (12)$$

In addition to scatter from measurement errors, it is sensible to assume that an *intrinsic* dispersion is also present³. From a theoretical point of view, this might be interpreted as either a lack of exact definition of the variable p which accounts for a linear relation, and/or the physics of galaxies requires actually additional variables for providing the absolute magnitude (i.e., the randomness interprets as the effect of these missing variables). Hence, we easily understand that a specific approach for defining $F(M, p)$ would require to presume a priori the related (unique) physical process, which is unfortunately not yet known⁴.

It must be noted that the goal is not to fit the data to such a model but to derive the Hubble Constant H_0 . Hence, in order to perform the ML technique, we ask whether we may substitute $F(M, p)$ by a suitable function which imitates it in reproducing the M - p correlation. The next section shows that the usual approaches, which consists of using the DTF and the ITF relations, interpret as a matter of fact as a particular choice of such a function.

It is advantageous to express a “linear correlation” by using a random variable of zero mean, which accounts for the data dispersion about the straight line (Δ_{TF}). In this purpose, it seems natural to use the “regular” distance, which is given by the segment orthogonal to this line. However, with our theoretical approach, it is equivalent (and suggested by arguments of simplicity) to use the following random variable

$$\zeta = \tilde{M} - M, \quad (13)$$

where \tilde{M} is given in Eq. (12), which is proportional to the regular distance (which reads $\zeta \cdot \cos(\arctan a)$). According to Eq. (12,13), obvious calculations show that, regardless of selection effects, the standard deviation σ_ζ of the ζ -distribution verifies

$$\sigma_\zeta^2 = (a\sigma_p - \rho_{\text{th}}(p, M)\sigma_M)^2 + (1 - \rho_{\text{th}}^2(p, M))\sigma_M^2, \quad (14)$$

where $\rho_{\text{th}}(p, M)$ denotes the theoretical correlation coefficient, see (Def.5). It is clear that the existence of an efficient correlation is expressed by the following inequality

$$\gamma = \frac{\sigma_\zeta}{\sigma_M} \ll 1, \quad (15)$$

or equivalently by $\sigma_\zeta \ll |a|\sigma_p$. Namely, Eq. (15) insures that p provides a good estimate of the absolute magnitude M from Eq. (12).

In order to proceed with the ML technique, we must presume the form of the ζ -pdf, hereafter denoted by $g(\zeta; 0, \sigma_\zeta)$, see (Def.1), which mimics the data distribution. In practice, a simple regression analysis should help us to guess the candidate

³The practical argument in favour of this approach is that presently the TF diagram is still continuously improved by shrewd corrections (as the galaxy orientation with respect to the line of sight, the linear dimension of the galaxy region from which the line width is measured, etc...), and thus that the M estimate given in Eq. (12) is approximate.

⁴One guesses that this lack of information is mainly responsible for the present debate on the choice between the DTF and ITF relations, an issue that is clarified below.

form to be used⁵. Now, a second random variable is required for specifying entirely the data distribution, let be ξ . For a trusty description of the TF diagram, the choice of ξ should be suggested by the appearance of the M - p distribution. Thus, in absence of such information, our working hypotheses for describing the M - p correlation are :

- h_3) the TF diagram can be mimic by the pdf

$$f_\xi(\xi)d\xi g(\zeta; 0, \sigma_\zeta)d\zeta \approx F(M, p)dMdp, \quad (16)$$

with a Gaussian ζ -distribution

$$g(\zeta; 0, \sigma_\zeta) = g_G(\zeta; 0, \sigma_\zeta), \quad (17)$$

see (Def.1.a).

The model defined by the pdf given by Eq. (16) is general enough to interpret the ITF and DTF methods used in the literature. Namely, the ITF relation $p = a_{\text{I}}M + b_{\text{I}}$, and the DTF relation $M = a_{\text{D}}p + b_{\text{D}}$, correspond to the following identifications :

$$\xi = \begin{cases} M & \text{in the ITF model} \\ p & \text{in the DTF model.} \end{cases} \quad (18)$$

Let us mention that such a definition has the advantage of avoiding a confusion which is inherent to usual approaches, e.g., see Teerikorpi (1990). Indeed, Eq. (16) tells us that ξ and ζ are uncorrelated outcomes, thus the use of conditional probabilities allows us to estimate a value of p from a value of $M = \xi$, in the case of ITF model, and we have the reverse situation in the case of DTF model, see Eq. (12,13). On the other hand, as long as the related random processes are not specified, the usual definitions would wrongly suggest that $a_{\text{I}} = 1/a_{\text{D}}$ and $b_{\text{I}} = -b_{\text{D}}/a_{\text{D}}$.

3. The technique of fitting

In the following, we proceed as follows : for each model, as defined by Eq. (16-18), we investigate the *calibration* of the TF relation (Step 1), see Eq. (12), and the *determination* of H_0 (Step 2).

In order to establish the likelihood function, the pdf is written in terms of observables, see (Def.4). The definition of these variables depends on the step of the analysis :

- Step 1) For the calibration of the TF relation, the data sample corresponds to the following observables

$$\{\mu_k, p_k, M_k = m_k - \mu_k\}_{k=1, N_1}, \quad (19)$$

see Eq. (7).

- Step 2) For the determination of the Hubble constant, it is more convenient to use the following observables

$$x = m - \eta, \quad (20)$$

$$y = a.p + b + \eta - m, \quad (21)$$

see Eq. (3,7), and the data sample corresponds to

$$\{\eta_k, x_k, y_k\}_{k=1, N_2}. \quad (22)$$

We use the following notations : N_1 , for the sample size, $\langle \rangle_1$, for the sample average, Cov_1 , for the sample covariance, etc... , in Step 1, while N_2 , $\langle \rangle_2$, Cov_2 , etc... , in Step 2, see (Def.5), which helps us to disentangle the statistics involved in each step. We must be aware that the random variables y and ζ (thus σ_ζ) are model dependent (through the estimates of a and b), while they are uniquely defined by Eq. (12,13,20).

⁵It is obvious that it must not be exponential, in order to ensure an effective M - p correlation.

3.1. Regardless of the Tully-Fisher relation

For reasons that appear clear in the following, let us derive the ML estimator of the Hubble constant when the TF-relation is ignored. The related statistical model is obtained by integrating the pd given by Eq. (4) over the variable p . According to Eq. (5), we obtain a theoretical pd which reads

$$\propto f_M(M; M_0, \sigma_M) dM \kappa(\mu) d\mu. \quad (23)$$

By writing this pd in terms of observables x and η , see Eq. (2,20), we easily understand that, for deriving an H_0 estimator, we must specify the $pdfs$ $f_M(M; M_0, \sigma_M)$ and $\kappa(\mu)$. Let us assume :

- h_4) a Gaussian luminosity distribution function,

$$f_M(M; M_0, \sigma_M) = g_G(M; M_0, \sigma_M); \quad (24)$$

and hypothesis (h_2), i.e., a uniformly spatial distribution, see Eq. (11). Hence, the lf is given by

$$\mathcal{L}(H_0) = -\ln P_{th}(\phi_m) - \beta\mathcal{H} - \frac{1}{N_2} \sum_{k=1}^{N_2} \frac{(x_k - M_0 + \mathcal{H})^2}{2(\sigma_M)^2}, \quad (25)$$

see (Def.4.b). Obvious calculations show that the normalization factor does not depend on H_0 as long as the selection effects are free of velocity criteria. Hence, the ML equation provides us with the following statistic

$$\mathcal{H}^C = (M_0 - \beta\sigma_M^2) - \langle x \rangle_2, \quad (26)$$

independently on whether the sample is complete up to a limiting magnitude. It is important to mention that, since $\partial P_{th}(\phi_m)/\partial \mathcal{H} = 0$, the term $\beta\sigma_M^2$ in Eq. (26) does not interpret as a Malmquist bias correction⁶, see (Def.6). In the case of η -selection effects, it is easy to show that the estimator (26) transforms by substituting β by $\beta + \partial \ln P_{th}(\phi_m \phi_\eta)/\partial \mathcal{H}$, where ϕ_η is the selection function describing the selection effects on velocities. In order to calculate the accuracy of Eq. (26), we need to specify the form of the selection function. For a magnitude limited complete sample, i.e., (h_1), see Eq. (9), the x -distribution reads $\propto g_G(x; M_0 - \mathcal{H} - \beta(\sigma_M)^2, \sigma_M)$, which shows that the standard deviation is equal to

$$\sigma_{\mathcal{H}^C} = \frac{\sigma_M}{\sqrt{N_2}}. \quad (27)$$

It is obvious that more accuracy is expected when the TF relation is used.

3.2. The inverse Tully-Fisher relation

Now we take into account the TF relation, by using the pdf defined by Eq. (16-18). The ITF model is specified by the choice $\xi = M$, which means that the random variables M and ζ are not correlated⁷. Namely, the data distribution in the M - p plane is supposed to be described by the following pd

$$f_M(M; M_0, \sigma_M) dM g_G(\zeta; 0, \sigma_\zeta) d\zeta \approx F(M, p) dM dp. \quad (28)$$

Let us remind that the precise rule of this pd is to mimic the data distribution, without interpreting the physical process involved in the TF diagram. This explains the respective locations of terms with respect to the equal sign in Eq. (28). The calculations, given in Appendix B, provide us with the following results :

Step 1) The ML equations yield statistics of $a \approx a^{ITF}$, $b \approx b^{ITF}$ and $\sigma_\zeta \approx \sigma_\zeta^{ITF}$, which are defined as follows

$$a^{ITF} = \frac{(\Sigma_1(M))^2}{\text{Cov}_1(p, M)}, \quad (29)$$

$$b^{ITF} = \langle M \rangle_1 - a^{ITF} \langle p \rangle_1, \quad (30)$$

$$\sigma_\zeta^{ITF} = \Sigma_1(M) \sqrt{\frac{1}{\rho_1^2(p, M)} - 1}. \quad (31)$$

It is interesting to note that, regarded as conventional estimators, these statistics show no correction for the Malmquist bias. Let us emphasize that they are valid for any form of the selection function ϕ_m . We easily understand that such a feature is of particular interest because a smooth decreasing function describes the selection effects on apparent magnitude more realistically than a sharp cutoff, as it is assumed by hypothesis (h_1). Moreover, because of the same reasons, it turns out that these statistics still work whatever the forms of functions f_M and κ , i.e., for any type of luminosity and spatial distributions of sources⁸. The (mathematical) reason of such properties is that the normalization factor $P_{th}(\phi)$ does not depend on model parameters a , b and σ_ζ , which is the case when the selection function reads $\phi = \phi_m$ or $\phi = \phi_m \phi_\mu$, see Appendix B.

Step 2) It turns out that the form of the ITF pdf , as given by Eq. (28) (see also Eq. (B8)), allows us to derive straightforwardly a first estimator, which is given by

$$\mathcal{H}^{ITF*} = \langle y \rangle_2. \quad (32)$$

It provides us with H_0 within the standard deviation

$$\sigma_{\mathcal{H}^{ITF*}} = \frac{\sigma_\zeta^{ITF}}{\sqrt{N_2}}, \quad (33)$$

see Eq. (29). Let us emphasize that it is obtained by preserving the above advantages, i.e., without assumptions on the completeness of the sample, the spatial and luminosity distributions.

On the other hand, the derivation of the ML statistics forces us to specify the functions κ and f_M . Thus, in addition of hypotheses (h_2, h_3), see Eq. (11,16), we assume a Gaussian luminosity distribution function (h_4), see Eq. (24). Hence, we obtain the following H_0 statistic

$$\mathcal{H}^{ITF} = \frac{\mathcal{H}^{ITF*} + \gamma^2 \mathcal{H}^C}{1 + \gamma^2}, \quad (34)$$

where $\gamma = \gamma^{ITF}$, see Eq. (15,26,32). The accuracy of such an estimate is calculated by specifying the function ϕ_m . By assuming (h_1), we obtain a standard deviation of

⁸Note that the pdf κ might simultaneously describe the selection effects on distance.

⁶While the sample average of absolute magnitudes in a magnitude limited sample is indeed biased because of the Malmquist effect, it is given by $\langle M \rangle_2 = M_0 - \beta\sigma_M^2$ under hypotheses (h_1, h_2, h_4).

⁷Actually, these variables may not be necessarily independent, e.g., the ζ - pdf may be Gaussian with a M -dependent standard deviation, $g(\zeta; 0, \sigma_\zeta(M))$, while Eq.(5,28) are still fulfilled.

$$\sigma_{\mathcal{H}^{\text{ITF}}} = \frac{\sigma_{\mathcal{H}^{\text{ITF}^*}}}{\sqrt{1 + \gamma^2}}. \quad (35)$$

It is obvious that Eq. (34) can be interpreted as a weighted mean of H_0 estimators, where the weighting factors correspond to related accuracies. Equation (35) shows that the ITF estimate is more accurate than the ITF* one. Hence, we understand that the advantage of having less constraints on the validity domain of the ITF* estimator (i.e., to have weak working hypotheses for defining the ITF* model), is to the detriment of the accuracy of estimates.

It turns out that both estimators \mathcal{H}^{ITF} and $\mathcal{H}^{\text{ITF}^*}$ can be used even when the sources are selected upon velocity criteria. The reason is that the related selection function $\phi_m(x + \eta)\phi_v(\eta)$, does not disturb the independence of y with respect to variables η and x , see Eq. (B8). On the other hand, these statistics become ineffective when selection rules are based on p , because $P_{\text{th}}(\phi_m, \phi_p)$ depends as a matter of fact on a and b .

It is interesting to note that little algebra allows us to write Eq. (34) as

$$\mathcal{H}^{\text{ITF}} = \frac{(\langle y \rangle_2 - \beta (\sigma_{\zeta}^{\text{ITF}})^2) + \gamma^2 (M_0 - \langle x \rangle_2)}{1 + \gamma^2}. \quad (36)$$

While providing the same quantity as Eq. (34), Eq. (36) is a different weighted mean of two quantities which are not H_0 estimators. The interpretation of this formulae is enlightened in Section 5.

3.3. The (direct) TF relation

The underlying working hypothesis used in the DTF approach is that the random variables p and ζ are not correlated. Namely, the data distribution in the M - p plane is supposed to be described by the following pd

$$f_p(p; p_0, \sigma_p) dp \, g_{\zeta}(\zeta; 0, \sigma_{\zeta}) d\zeta \approx F(M, p) dM dp. \quad (37)$$

It turns out that this approach forces us to presume a priori the form of functions $\phi_m(m)$, $f_p(p; p_0, \sigma_p)$ and $\kappa(\mu)$. Hence, we use (h_1, h_2, h_3) , see Eq. (9,11,16), and for reason of coherence with h_4 , we assume :

- h'_4) a Gaussian p -distribution,

$$f_p(p; p_0, \sigma_p) = g_G(p; p_0, \sigma_p). \quad (38)$$

The calculations, which are given in Appendix C, provide us with the following results :

Step 1) The likelihood equations yield statistics of $a \approx a^{\text{DTF}}$, $b \approx b^{\text{DTF}}$ and $\sigma_{\zeta} \approx \sigma_{\zeta}^{\text{DTF}}$, which are defined as follows :

$$a^{\text{DTF}} = \frac{\text{Cov}_1(p, M)}{(\Sigma_1(p))^2}, \quad (39)$$

$$b^{\text{DTF}} = \langle M \rangle_1 - a^{\text{DTF}} \langle p \rangle_1 + \beta (\sigma_{\zeta}^{\text{DTF}})^2, \quad (40)$$

$$\sigma_{\zeta}^{\text{DTF}} = \Sigma_1(M) \sqrt{1 - \rho_1^2(p, M)}. \quad (41)$$

It is interesting to note that, regarded as conventional statistics, only the estimator given in Eq. (39) shows a correction for a bias.

Step 2) The ML statistic of H_0 is given by

$$\mathcal{H}^{\text{DTF}} = \langle y \rangle_2 - \beta (\sigma_{\zeta}^{\text{DTF}})^2, \quad (42)$$

and has a standard deviation equal to

$$\sigma_{\mathcal{H}^{\text{DTF}}} = \frac{\sigma_{\zeta}^{\text{DTF}}}{\sqrt{N_2}}, \quad (43)$$

see Eq. (39).

The above statistics are no longer valid when selection effects on p , and as well as on μ , are present. Nevertheless, they can easily be adapted by rewriting the p -pdf as $\tilde{f}_p(p) \propto \phi_p(p) f_p(p; p_0, \sigma_p)$, for taking into account p -selection effects. Let us emphasize that the correction in Eq. (42) is not of Malmquist type, in contrast with the one in Eq. (39), see Section 5. Moreover, it must be noted that the magnitude of the bias does not depend on the limiting magnitude m_{lim} , and of σ_p (or equivalently σ_M).

3.4. Comparison of estimators

It is important to understand that the Hubble constant H_0 has a similar status among these models, contrarily to parameters a and b which define the data distribution on the TF diagram. Indeed, it must be noted that the ITF model inherits all model parameters defined in the ITF* model, simply because it is a particular case, where the functions $\phi_m(m)$, $\kappa(\mu)$ and $f_M(M)$ are specified. Hence, it is clear that H_0 keeps an identical status. On the other hand, because the DTF model and the ITF models describe the data distribution in a different way (see Appendix D), we might expect to obtain a different status. However, let us note that the pd given in Eq. (23) is the projection of the ITF* pd , as well as the DTF one. Therefore, the model parameters which are in common (i.e., which are not cancelled by the projection), are identical among these models, which is the case of H_0 .

Therefore, according to previous sections, which show that Eq. (26,32,34,42) define unbiased H_0 statistic, the related estimates (for a given sample) correspond as a matter of fact to the same quantity, and the discrepancies (between these different estimates) interpret as statistical fluctuations which should vanish when the sample size increases. With this in mind, we investigate the nature of such discrepancies. These quantities can be derived from the following ones

$$\Delta_{\Pi^*} = \mathcal{H}^{\text{ITF}} - \mathcal{H}^{\text{ITF}^*}, \quad (44)$$

$$\Delta_{\text{CI}^*} = \mathcal{H}^{\text{C}} - \mathcal{H}^{\text{ITF}^*}, \quad (45)$$

$$\Delta_{\text{DI}^*} = \mathcal{H}^{\text{DTF}} - \mathcal{H}^{\text{ITF}^*}, \quad (46)$$

where the ITF* is chosen as a reference estimate. According to Eq. (44,46), the difference between the statistics given in Eq. (32,34) reads

$$\Delta_{\Pi^*} = \frac{\gamma^2}{1 + \gamma^2} \Delta_{\text{CI}^*} \quad (47)$$

where $\gamma = \gamma^{\text{ITF}}$, see Eq. (15). Thus, the smaller the ratio γ^{ITF} , the smaller the discrepancy Δ_{Π^*} . Let us elucidate this particular property, which shows clearly the gain of knowledge on H_0 when the TF relation is used. It is important to note that the estimates \mathcal{H}^{C} and $\mathcal{H}^{\text{ITF}^*}$, given in Eq. (26,47), are independent, i.e., they involve two different types of information. Indeed, the \mathcal{H}^{C} is based only on characteristics related to the luminosity distribution function of sources (or equivalently, on the p -distribution), while the $\mathcal{H}^{\text{ITF}^*}$ takes into account only

the TF relation (i.e., the ζ -distribution). When both features are used, we obtain a more accurate estimate \mathcal{H}^{ITF} , which lies between \mathcal{H}^{C} and $\mathcal{H}^{\text{ITF*}}$. Actually, it lies much more close to $\mathcal{H}^{\text{ITF*}}$, accordingly to Eq. (15), which shows that this estimator is less sensitive to hypotheses on the luminosity distribution function of sources, which translates the robustness of the estimator $\mathcal{H}^{\text{ITF*}}$.

In order to calculate $\Delta_{\text{DI*}}$, let us compare the statistics (29-29) and (39-39). After little algebra, we obtain

$$a^{\text{DTF}} = \rho_1^2(p, M) \times a^{\text{ITF}}, \quad (48)$$

$$b^{\text{DTF}} = b^{\text{ITF}} + (1 - \rho_1^2(p, M)) (a^{\text{ITF}} \langle p \rangle_1 + \beta(\Sigma_1(M))^2), \quad (49)$$

$$\sigma_{\zeta}^{\text{DTF}} = |\rho_1| \times \sigma_{\zeta}^{\text{ITF}}. \quad (50)$$

Hence, according to Eq. (29,32,42,44), it follows⁹

$$\Delta_{\text{DI*}} = \frac{\rho_1}{|\rho_1|} C_p \frac{\gamma}{\sqrt{1+\gamma^2}} \sigma_{\zeta}^{\text{ITF}} \quad (51)$$

where $\gamma = \gamma^{\text{ITF}}$, see Eq. (15), $\rho_1 = \rho_1(p, M)$ and

$$C_p = \frac{\langle p \rangle_1 - \langle p \rangle_2}{\Sigma_1(p)}. \quad (52)$$

Thus $\Delta_{\text{DI*}}$ depends essentially on two independent characteristics. The first one is the discrepancy of sample averages of p values between the calibration sample and the one used to determine H_0 . The second one is the accuracy of the TF relation, and similarly as above, Eq. (51) shows that the smaller the ratio γ^{ITF} the smaller the discrepancy. Let us emphasize that such a feature can also be interpreted in terms of M - p correlation, since we have $\gamma^2/(1+\gamma^2) \approx 1 - \rho_1(p, M)^2$, see Eq. (15,29). Thus, the higher the value of $|\rho_1(p, M)|$ the smaller the discrepancies given in Eq. (47,51). The hypothetical case $|\rho_1(p, M)| = 1$ is a singular situation, where the data distribution on the TF diagram coincides with the straight line (Δ_{TF}) defined by Eq. (12), which makes the ITF*, the ITF and DTF approaches identical.

Now let us calculate the expected orders of magnitude of discrepancies given in Eq. (44-44), and their dependence on the sample size. Note that the statistics defined in Eq. (26,32), considered as random variables (see Sec. B.2), are independent. Hence, $\Delta_{\text{CI*}}$, resp. $\Delta_{\text{II*}}$, both have a vanishing expected value, with a standard deviation

$$\sigma_{\Delta_{\text{CI*}}} \approx \sqrt{1+\gamma^2} \frac{\sigma_M}{\sqrt{N_2}}, \quad (53)$$

resp.

$$\sigma_{\Delta_{\text{II*}}} \approx \frac{\gamma}{\sqrt{1+\gamma^2}} \frac{\sigma_{\zeta}^{\text{ITF}}}{\sqrt{N_2}}. \quad (54)$$

Therefore, the difference between the ITF and the ITF* estimates is not systematic, and thus there is no bias. Moreover these estimates coincide as the sample size N_2 increases. Similarly, since $\langle p \rangle_1$, resp. $\langle p \rangle_2$, is a statistic providing the mean p_0 with a standard deviation of $\sigma_p/\sqrt{N_1}$, resp. $\sigma_p/\sqrt{N_2}$, and that *Step 1* and *Step 2* are independent, then C_p is a random variable of vanishing mean with standard deviation $\approx \sqrt{1/N_1 + 1/N_2}$. Hence, $\Delta_{\text{DI*}}$, has a vanishing expected value, with a standard deviation

⁹Let us remind that the average $\langle y \rangle_2$ is a model-dependent quantity.

$$\sigma_{\Delta_{\text{DI*}}} = \frac{\gamma}{\sqrt{1+\gamma^2}} \sigma_{\zeta}^{\text{ITF}} \sqrt{1/N_1 + 1/N_2}. \quad (55)$$

Thus there is no bias between the DTF and the ITF* estimates, while the estimates of model parameters a and b are different, see Eq. (48-48). Nevertheless, it must be noted that these estimates coincide only when both sample sizes N_1 and N_2 increase, which emphasizes the importance of the calibration of the TF relation.

Let us remind that the ITF and the DTF models belong to a single class of models defined by Eq. (16). Intermediate models can be obtained by means of a rotation parameter which makes a link between the ITF and the DTF models. Thus, we understand that simple arguments (of linearity) indicate that the H_0 statistics related to these models provide asymptotically identical estimates, Triay et al. (1993b). More generally, we might ask whether this is still true for models describing the TF diagram in a more complex way. The element of answer comes by noting that we have $y = \zeta + \mathcal{H}$, see Eq. (12,13,20), and that $P_{\text{th}}(y) = \mathcal{H}$ independently of hypothesis (h_3), which forces the random variable ζ to have a vanishing mean value. Thus, we can claim that if this condition is complete then the way of describing the data does not influence the determination of H_0 .

On account of these results, let us ask whether the accuracy of estimates may be used as criterion for having a preference for a particular model. According to Eq. (15,33,35, 43,48), it turns out that we obtain

$$\sigma_{\mathcal{H}^{\text{DTF}}} \approx \sigma_{\mathcal{H}^{\text{ITF}}}, \quad (56)$$

while the ITF* estimate is less accurate, see Eq. (35). Thus, we have similar precisions in estimating H_0 by using indifferently either the DTF or the ITF approaches. Work is in progress for checking whether an intermediate statistical model which describes the TF relation might provide higher accuracy on the determination of H_0 , Triay et al. (1993b).

According to Eq. (51), it is interesting to note that if the following equality

$$C_p = 0, \quad (57)$$

herein called “ C_p -criteria”, is fulfilled for a given sample, then the *algebraic* expressions of the ITF* and the DTF estimators of H_0 become identical. On the other hand, if the “ C_p -criteria” is not verified then nothing prevents us to resample (to remove data according to rules allowed by the working hypotheses) on this purpose (although it reduces the sample size, and thus diminish the information). Namely, the faintest objects can be removed until Eq. (57) is complete, which preserves the selection rule based on the magnitude selection effect (i.e., the hypothesis h_1), but with a brighter limiting magnitude.

3.5. Applications

In order to have a visual support for our theoretical approach and to investigate the influence of calibration errors in the determination of H_0 , we perform $N_S = 1000$ simulations. We generate two sorts of samples : the $\{m_k, \mu_k, p_k\}_{k=1, N_1}$, which is involved in the calibration step, and the $\{m_k, \eta_k, p_k\}_{k=1, N_2}$, which is involved in the determination of the Hubble constant H_0 , this one contains $N_2 = 100$ objects. For each sample, we

have three independent data sets : the absolute magnitudes $\{M_k\}$, the intrinsic TF dispersions $\{\zeta_k\}$, according to working hypotheses (h_3, h_4) , see Eq. (16,24), and the distance moduli $\{\mu_k\}$, according to working hypotheses (h_1, h_2) , see Eq. (11); the $\{p_k\}$ are derived from Eq. (12,13). The apparent magnitudes $\{m_k\}$ are given by Eq. (7), and the cosmological velocity moduli $\{\eta_k\}$ are calculated according to Eq. (3), with a Hubble constant given by

$$H_0^S = 100 \text{ km s}^{-1} / \text{Mpc}. \quad (58)$$

The characteristics of samples are the following :

- completeness of samples, up to a limiting magnitude of $m_{\text{lim}} = 12$; (59)
- a uniform spatial distribution of sources;
- a Gaussian luminosity distribution function, defined by $M_0 = -19$, $\sigma_M = 1.5$; (60)
- we assume (a priori) the ITF model, so that the TF diagram shows a normal ζ -dispersion at constant M defined by

$$\sigma_\zeta = 0.5, \quad (61)$$

and with the following calibration parameters

$$a = -6 \quad \text{and} \quad b = -7. \quad (62)$$

A first set of simulations is performed in order to investigate the H_0 statistics regardless of calibration errors. In this case, a unique calibration sample is used, the sample size of $N_1^{(1)} = 8000$ galaxies is large enough so that the estimates of model parameters a , b and σ_ζ , as given by Eq. (29-29,39-39), are expected to be free of statistical fluctuations. The re-

Table 1. Comparison of estimators. The estimates of parameters a , b and σ_ζ are obtained from a sample of $N_1^{(1)} = 8000$ galaxies (which gives values which are free of statistical fluctuations). The related standard deviations are obtained from 30 trials of such samples. The estimate of H_0 corresponds to the mean value on $N_S = 1000$ trials with samples of $N_2 = 100$ objects.

Parameter	ITF*	ITF	DTF
a	-5.99 ± 0.02	-5.39 ± 0.02	
b	-7.02 ± 0.04	-8.22 ± 0.05	
σ_ζ	0.500 ± 0.004	0.475 ± 0.003	
$H_0 - H_0^S$	-0.2 ± 2.4	-0.2 ± 2.2	-0.2 ± 2.2

sults are given in Table 1. It is reassuring to note that the ITF estimates of model parameters a , b and σ_ζ , correspond as a matter of fact to values used to generate the random samples, see Eq. (62), while it is not the case for the DTF estimates. These estimates are used to determine H_0 , according to Eq. (32,34,42), on $N_S = 1000$ samples of $N_2 = 100$ objects. The average of these estimates and their related accuracy (1σ) are given in Table 1. In agreement with the theory, we can note that these statistics give back the value H_0^S , used for the simulations, which shows that they are not biased. Moreover, we can note that the related accuracies are in agreement with the expected value obtained from Eq. (33,35,43), where we use $\sigma_{\mathcal{H}} \approx \Sigma(\mathcal{H})$. Figure 1 shows 100 H_0 -estimates from these dif-

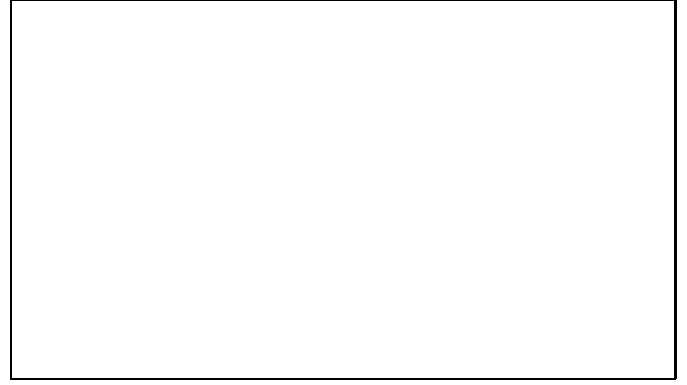


Fig. 1. Comparison between the ITF, ITF* and the DTF estimates, without taking into account calibration errors. The x -coordinate of symbols correspond to ITF estimates. The y -coordinate of symbols “+” correspond to DTF estimates, while the y -coordinate of symbols “O” correspond to ITF* estimates.

ferent approaches. The symbols “+” correspond to the DTF versus the ITF estimate, and the symbols “O” correspond to the ITF* versus the ITF estimate. The evident distribution along the diagonal shows that these approaches are as a matter of fact equivalent. Moreover, we can speculate that it is advantageous to prefer the ITF* approach, which is more robust, since there is no real gain of accuracy by choosing the other ones. It is interesting to note that the distribution of symbols “O” is more scattered about the diagonal than the symbols “+”. The differences of accuracy between these estimates don’t suffice to account for such a gap (which can be estimated of the order of 0.01). Therefore, this means that the ITF*-ITF estimates are less correlated than the DTF-ITF ones, while one might expect the opposite. Indeed, let us remind that the ITF model is defined from the ITF* model by specifying the functions ϕ_m , κ and f_M , and thus it is simply a particular case of the ITF* model, while it is different from the DTF model. The reason lies in the amount of information which is used by these estimators. Indeed, the ITF* estimator uses less working hypotheses than the ITF and the DTF ones, which makes it more slacker.

Table 2. Effects due to calibration errors. These estimates correspond to mean values calculated on $N_S = 1000$ trials. The parameters a , b and σ_ζ are obtained from samples of $N_1^{(2)} = 30$ galaxies, while H_0 is determined from samples of $N_2 = 100$ objects.

Parameter	ITF*	ITF	DTF
a	-6.06 ± 0.40	-5.40 ± 0.32	
b	-6.84 ± 1.02	-8.21 ± 0.85	
σ_ζ	0.498 ± 0.075	0.469 ± 0.064	
$H_0 - H_0^S$	-0.2 ± 4.4	-0.2 ± 4.0	-0.1 ± 4.0

In general, the calibration of the TF relation is performed only on few tens of galaxies, which makes the estimates of model parameters a , b and σ_ζ much less precise. Hence, the determination of H_0 undergoes the related statistical fluctuations, herein called *calibration errors*. So such effects are inves-

tigated by using simulated calibration samples, with a more realistic sample size of $N_1 = 30$ galaxies, and by determining H_0 on samples of $N_2 = 100$ objects. The statistical analysis is performed on $N_S = 1000$ trials. The results are shown in Table 2, which gives the averages of parameters estimates, and their related accuracy (1σ). The comparison with Table 1 shows that the estimates of model parameters a , b and σ_ζ are similar, and that the estimation of H_0 is not biased by calibration errors, while it is obviously less accurate. Figure 2 shows more



Fig. 2. Comparison between the ITF, ITF* and the DTF estimates, by taking into account calibration errors. The same caption as in Fig. 1.

elongated distributions than those in Fig. 1, but still about the diagonal. At first glance, the main result is that the above conclusions are still valid when calibration errors are taken into account (actually, it is easy to note that these approaches are even much more equivalent, since the scatter of symbols “+” about the diagonal is now comparable to the one of symbols “○”).

4. About measurement errors

It is clear that Malmquist bias is present only when a part of the (luminosity) *pdf* is not observed. Since this is independent of errors distribution, we understand that measurement errors do not produce such a bias, see (Def.6). Notwithstanding, in order to answer the question of whether these effects introduce another type of bias, we must use a precise mathematical framework for avoiding misunderstandings.

4.1. The statistical model

Let $\{\epsilon_m, \epsilon_\mu, \epsilon_p, \epsilon_\eta\}$ denote the measurement errors. We use the symbol hat “ $\hat{\cdot}$ ” to distinguish the (*measurable*) variables (i.e. the ones which are affected by measurement errors), from formal ones which are given by

$$m = \hat{m} - \epsilon_m, \quad (63)$$

$$\mu = \hat{\mu} - \epsilon_\mu, \quad (64)$$

$$p = \hat{p} - \epsilon_p, \quad (65)$$

$$\eta = \hat{\eta} - \epsilon_\eta. \quad (66)$$

If these errors are independent Normal random variables, they are distributed according to the following the *pd*

$$dP_\epsilon^{(s)} = \prod_{\lambda \in \Lambda^{(s)}} g_G(\epsilon_\lambda; 0, \sigma_{\epsilon_\lambda}^{(s)}) d\epsilon_\lambda, \quad (67)$$

where the index λ takes character values among the set $\Lambda^{(1)} = \{m, \mu, p\}$ or $\Lambda^{(2)} = \{m, \eta, p\}$, depending on *Step 1* or *Step 2*. Since a magnitude limited sample can be selected from a catalog, where the data are already affected by measurement errors, we easily understand that the selection function must be written in term of measurable observables. Therefore, according to Eq (8), the *pd* which describes both the observables and the random errors is given by

$$d\hat{P}_{\text{obs}}^{(s)} = \frac{\hat{\phi}_m}{P_{\text{th}} \left(P_\epsilon^{(s)} (\hat{\phi}_m) \right)} dP_{\text{th}} \times dP_\epsilon^{(s)}, \quad (s = 1, 2), \quad (68)$$

where

$$\hat{\phi}_m(m) = \phi_m(\hat{m}). \quad (69)$$

It is clear that since the errors ϵ_λ , see Eq. (63-63), cannot be disentangled from intrinsic scatter, the ML technique is not feasible for obtaining genuine statistics. However, we can overcome this obstacle by substituting the suitable corrections by their expected values, which can be calculated according to the *pd* given in Eq. (68). For convenience in writing, we use the following dimensionless quantities

$$\delta_p = \frac{(\sigma_{\epsilon_p}^{(1)})^2}{(\Sigma_1(\hat{p}))^2}, \quad (70)$$

$$\delta_M = \frac{(\sigma_{\epsilon_m}^{(1)})^2 + (\sigma_{\epsilon_\mu}^{(1)})^2}{(\Sigma_1(\hat{M}))^2}, \quad (71)$$

see Eq. (7), and $\hat{\mathcal{C}}_p$, see Eq. (52). It turns out that the spatial distribution of sources must be specified (i.e., $\kappa(\mu)$) a priori in order to perform such calculations, see Appendix E. If we assume that it is uniform (h_2), see Eq. (11), then the normalization term is given by

$$P_{\text{th}} \left(P_\epsilon^{(s)} (\hat{\phi}_m) \right) = P_{\text{th}} (\phi_m) \exp \left(\frac{1}{2} (\beta \sigma_{\epsilon_m}^{(s)})^2 \right), \quad (72)$$

where $P_{\text{th}} (\phi_m)$ is the normalization term when measurement errors are not taken into account. The effects due to measurement errors lie only in the extra term, which turns out to be independent on model parameters, and thus which ensures the absence of Malmquist bias on estimating these parameters. Nevertheless, there are biases of different nature in the H_0 estimates, which are given in Eq. (26,32,42), since the calculation provides us with

$$\begin{aligned} \mathcal{H}^{\text{ITF}^*} &= \hat{\mathcal{H}}^{\text{ITF}^*} + \hat{\mathcal{C}}_p \delta_M \hat{a}^{\text{ITF}} \Sigma_1(\hat{p}) \\ &+ \beta \left((\sigma_{\epsilon_m}^{(1)})^2 - (\sigma_{\epsilon_m}^{(2)})^2 \right), \end{aligned} \quad (73)$$

$$\mathcal{H}^C = \hat{\mathcal{H}}^C + \beta (\sigma_{\epsilon_m}^{(2)})^2, \quad (74)$$

$$\begin{aligned} \mathcal{H}^{\text{DTF}} &= \hat{\mathcal{H}}^{\text{DTF}} + \hat{\mathcal{C}}_p \frac{\delta_p}{1 - \delta_p} \hat{a}^{\text{DTF}} \Sigma_1(\hat{p}) \\ &+ \beta \left((\sigma_{\epsilon_m}^{(1)})^2 - (\sigma_{\epsilon_m}^{(2)})^2 \right). \end{aligned} \quad (75)$$

The bias free ITF statistics is obtained by substituting in Eq.(34) the terms $\mathcal{H}^{\text{ITF}^*}$ and \mathcal{H}^C , as given by Eq.(73,73). Therefore, we see that the statistics given in Eq. (26,32,34,42)

can be restored as long as the standard deviations of errors are known, i.e., σ_{ϵ_m} and σ_{ϵ_μ} in the case of the ITF model, and σ_{ϵ_p} in the case of the DTF model¹⁰. However, it is clear that these corrections should be tiny quantities, since $\delta_p \ll 1$ and $\delta_M \ll 1$, unless the information is buried into noise. Moreover, it is interesting to note that they are of different nature, the first one depends on TF characteristics and can be removed by using the C_p -criteria, see Eq. (57), while the other one does not.

4.2. Applications

Similarly to Section 3.5, we perform simulations in order to enlighten on above results and to investigate the effects of measurement errors on the accuracy of estimates. According to working hypotheses, we use simulated samples with characteristics given by Eq. (58-62), where the observables are perturbed by normal random errors. In practice, according to Gouguenheim (1993), the observables are measured within the following accuracies :

- the line width (which gives p) is measured within 20 km/s;
- the recession of galaxies is given within 15 km/s;
- for calibrators (*Step 1*), the apparent magnitudes are measured within an accuracy of 0.05 mag., while for *Step 2*, the accuracy depends on magnitude, it is of order of 0.1 mag. for $m \leq 13$, of 0.15 mag. for $13 < m \leq 14$, and of 0.2 mag. for $m > 14$.

These above uncertainties can be interpreted as the 2-3 errors standard deviations. We can use a good compromise on the magnitude of errors for avoiding their dependence on the magnitude of the related variable by assuming the following characteristics :

$$\sigma_{\epsilon_m}^{(1)} = 0.05, \quad \sigma_{\epsilon_p}^{(1)} = 0.025, \quad \sigma_{\epsilon_\mu}^{(1)} = 0.15, \quad (76)$$

and for samples used to determine H_0 , according to Fouqué (1993), we choose

$$\sigma_{\epsilon_m}^{(2)} = 0.15, \quad \sigma_{\epsilon_p}^{(2)} = 0.025, \quad \sigma_{\epsilon_\eta}^{(2)} = 0.0. \quad (77)$$

The effect on H_0 statistics due to measurement errors, but regardless of calibration errors, is investigated by using a unique calibration sample with $N_1^{(1)} = 8000$ galaxies, and a sample of $N_2 = 100$ objects for the H_0 determination. These samples are generated according to Eq. (58-62), and both are perturbed by normal random errors with characteristics defined by Eq. (76,77). The statistical analysis is performed on $N_S = 1000$ trials. The statistics of model parameters a , b and σ_ζ , corrected for the bias due to measurement errors are defined in Eq. (E14-E17). The related results are given in Table 3, which shows the averages of model parameters estimates and their related accuracies (1σ), and the magnitude of the correction terms which are present in the H_0 statistics, given in Eq. (73-73). We can note that these corrections are effective since the mean value of H_0 estimates gives back the value H_0^S . However, it is clear that this is a minor quantity compared to the H_0 standard deviation. Figure 3 shows the same diagram as in Fig. 1. The comparison between these figures indicates that the measurement errors do not perturb the correlation between the ITF*, the ITF and the DTF methods. A similar analysis

¹⁰Note that these criteria can motivate the choice of the model to be used.

Table 3. Measurement errors, without calibration errors. The results are based on $N_S = 1000$ trials. The parameters a , b and σ_ζ , obtained from samples of $N_1^{(1)} = 8000$ galaxies, while H_0 is measured from samples of $N_2 = 100$ objects. The correction term $\Delta_\epsilon H_0$ is written in H_0 unit.

Parameter	ITF*	ITF	DTF
a	-6.00 ± 0.03		-5.40 ± 0.02
b	-6.99 ± 0.07		-8.20 ± 0.06
σ_ζ	0.501 ± 0.005		0.475 ± 0.004
$H_0 - H_0^S$	-1.2 ± 2.5	-1.2 ± 2.4	-1.2 ± 2.4
$\Delta_\epsilon H_0$	-1.27 ± 0.08	-1.27 ± 0.07	-1.27 ± 1.79

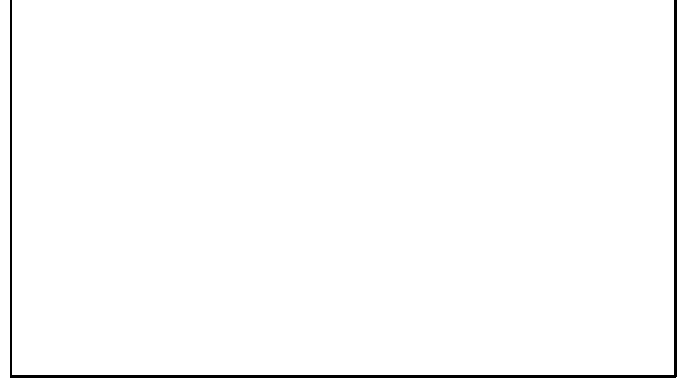


Fig. 3. Comparison between the ITF, ITF* and the DTF estimates. The calibration error are taken into account. The same caption as in Fig. 1.

is performed by taking into account simultaneously calibration errors. The method of proceeding is identical to Section 3.5 above. The results are given in Table 4, and in Fig. 4. The main effect of calibration errors is to increase the standard deviation of both the correction term and H_0 , which does not change our previous conclusions, while the ITF estimate seems to be 10 percent more accurate.

Table 4. Measurement errors, with calibration errors. The same caption as in Table 3.

Parameter	ITF*	ITF	DTF
a	-6.03 ± 0.42		-5.43 ± 0.35
b	-6.90 ± 1.06		-8.12 ± 0.90
σ_ζ	0.482 ± 0.084		0.457 ± 0.073
$H_0 - H_0^S$	-0.6 ± 5.2	-0.6 ± 4.8	-0.8 ± 5.0
$\Delta_\epsilon H_0$	-1.29 ± 0.13	-1.29 ± 0.11	1.25 ± 1.77

5. The distances of galaxies and H_0

Within the same framework, let us investigate the problem of finding a reliable determination of distances of galaxies, by using the TF relation. This still involves a first step of calibration of the TF relation, and thus we refer to above results. On the



Fig. 4. Comparison between the ITF, ITF* and the DTF estimates. The calibration errors are taken into account. The same caption as in Fig. 1.

other hand, the next step does not involve a second sample, but only a unique galaxy, whose the related data are denoted by m_k and p_k . At first glance, according to Eq. (7,12), the distance modulus of a galaxy of known apparent magnitude m_k and line width distance estimator p_k , can be estimated by the following statistic

$$\tilde{\mu}_k = m_k - (a.p_k + b). \quad (78)$$

We easily understand that, because a and b are (as a matter of fact) model dependent parameters, similarly to the above approach, the question of whether this (ad hoc) estimate is biased takes its answer only within a given model. Thus we have to define a pd which describes the distribution of distance modulus μ provided $m = m_k$ and $p = p_k$, that we denote $dP_\mu^{(k)} = f_\mu(\mu; \mu_0^{(k)}, \sigma_\mu^{(k)})d\mu$. According to Eq. (8), by using conditional probabilities, we have

$$dP_\mu^{(k)} = \frac{\delta(m - m_k)\delta(p - p_k)}{P_{\text{obs}}(\delta(m - m_k)\delta(p - p_k))} dP_{\text{obs}}, \quad (79)$$

and thus

$$f_\mu(\mu; \mu_0^{(k)}, \sigma_\mu^{(k)}) = \frac{F(m_k - \mu, p_k) \kappa(\mu)}{\int F(m_k - \mu, p_k) \kappa(\mu) d\mu}. \quad (80)$$

Since no selection function intervenes in this equation, we deduce that no Malmquist bias is present. Moreover, the expected value of the distance modulus, which is given by

$$\mu_0^{(k)} = \int \mu dP_\mu^{(k)}, \quad (81)$$

see Eq. (79), provides us with the most likelihood statistic. It is clear that such an estimate depends on the form of the $TFpdf$, which reads

$$F(m_k - \mu, p_k) = g_G(\mu; \tilde{\mu}_k, \sigma_\zeta) \times \begin{cases} a f_M(m_k - \mu; M_0, \sigma_M) & \text{(ITF)} \\ f_p(p_k; p_0, \sigma_p) & \text{(DTF)} \end{cases} \quad (82)$$

see Eq. (28,37, 81). Hence, the pd given in Eq. (79), transforms

$$dP_\mu^{(k)} \propto g_G(\mu; \tilde{\mu}_k, \sigma_\zeta) \kappa(\mu) d\mu \times \begin{cases} f_M(m_k - \mu; M_0, \sigma_M) & \text{(ITF)} \\ 1 & \text{(DTF)} \end{cases} \quad (83)$$

The very interesting feature, which is shown by Eq. (83), is that the distance modulus estimate given by the DTF model, see Eq. (81), does not depend on the luminosity distribution function of sources. Secondly, (in all cases) the distance modulus estimate given in Eq. (78) must be corrected for a bias. In order to calculate the correction term, we have to specify the luminosity distribution function $f_M(M)$ in the case of the ITF model, and the function $\kappa(\mu)$ in both cases. So let us assume that the sources are uniformly distributed in space (h_2), and eventually that they show a Gaussian luminosity distribution function (h_4). Hence, by using (Def.2.c.d), we obtain straightforwardly the following distance modulus estimate

$$\mu_0^{(k)} = \begin{cases} \frac{1}{1+\gamma^2} ((\tilde{\mu}_k + \beta\sigma_\zeta^2) + \gamma^2(m_k - M_0)) & \text{(ITF)} \\ \tilde{\mu}_k + \beta\sigma_\zeta^2 & \text{(DTF)} \end{cases}, \quad (84)$$

where $\gamma = \gamma^{\text{ITF}}$, see Eq. (15), with a (accuracy) standard deviation $\sigma_\mu^{(k)} = \sigma_\mu$,

$$\sigma_\mu = \begin{cases} \sigma_\zeta / \sqrt{1 + \gamma^2} & \text{(ITF)} \\ \sigma_\zeta & \text{(DTF)} \end{cases} \quad (85)$$

According to Eq. (15,29,48), it turns out that these distance modulus estimates have similar accuracy, while information is used for calculating the ITF distance modulus estimate. By using Eq. (48-48), we easily calculate the difference $\Delta\mu = \mu^{\text{ITF}} - \mu^{\text{DTF}}$ between the distance modulus estimates. We obtain

$$\Delta\mu = \frac{\gamma^2}{1 + \gamma^2} (a^{\text{ITF}} \langle p \rangle_1 + b^{\text{ITF}} + \beta(\Sigma_1(M))^2 - M_0). \quad (86)$$

Let us emphasize that $a^{\text{ITF}} \langle p \rangle_1 + b^{\text{ITF}} + \beta(\Sigma_1(M))^2$ is an unbiased statistics which gives M_0 within $\sigma_M / \sqrt{N_1}$. Therefore, Eq. (86) shows that the difference turns out to be a tiny quantity. Namely, $\Delta\mu$ has a vanishing expected value, with a standard deviation given by

$$\sigma_{\Delta\mu} = \frac{\sigma_\zeta^{\text{ITF}}}{\sqrt{N_1}} \frac{\gamma}{\sqrt{1 + \gamma^2}}, \quad (87)$$

where $\gamma = \gamma^{\text{ITF}}$, see Eq. (15). Therefore, this shows that we obtain the same distance modulus estimate by using different models.

Finally, we come to the conclusion that the choice of the model should be based on the reliability of hypotheses used about the selection effects, and it is interesting to note that the DTF approach is more robust than the ITF one. The correction terms in Eq. (84) are not related to biases of Malmquist type but identify to *volume corrections*, herein calculated for *homogeneous* spatial distributions or of power law type (i.e., $\beta \neq \frac{3 \ln 10}{5}$), the *inhomogeneous* case is straightforward.

Obvious calculations show that the effects of calibration errors on distance modulus estimates make them less accurate by introducing a white noise of a p -dependent standard deviation given by

$$\sigma_\mu^{\text{cal}} = \sqrt{\sigma_{\delta_a}^2 p^2 - 2\text{Cov}(\delta_a, \delta_b)p + \sigma_{\delta_b}^2}, \quad (88)$$

where δ_a and δ_b denote the calibration errors. The simulations show that

$$\sigma_{\delta_a} = \begin{cases} 0.21 & (\text{ITF}) \\ 0.22 & (\text{DTF}) \end{cases} \quad (89)$$

$$\sigma_{\delta_b} = \begin{cases} 0.53 & (\text{ITF}) \\ 0.56 & (\text{DTF}) \end{cases} \quad (90)$$

$$\text{Cov}(\delta_a, \delta_b) = \begin{cases} -0.11 & (\text{ITF}) \\ -0.12 & (\text{DTF}) \end{cases} \quad (91)$$

Now it is natural to make the link between the distance modulus and the H_0 estimates. A simple formal comparison between Eq. (34,42,84), provides us with

$$\frac{1}{N_2} \sum_{k=1}^{N_2} (\eta_k - \mu_0^{(k)}) = \begin{cases} \mathcal{H}^{\text{ITF}} & (\text{ITF}) \\ \mathcal{H}^{\text{DTF}} & (\text{DTF}) \end{cases} \quad (92)$$

It is obvious that such an equality is valid within the common set of hypotheses, which confines to those specified for Eq. (34,42,84). Now, we can understand that the H_0 statistics given in Eq. (36) has its foundation in a context of distance estimates.

6. Conclusion

We present a general framework to estimate the Hubble constant, as well as the distances of galaxies, when their peculiar velocities are neglected, by using distance estimators given by the Tully-Fisher, or the Faber-Jackson relations. Such relations can be regarded as a single law describing the observed linear correlation between the absolute magnitude M of galaxies and their line width distance indicator p . This well known problem has been enlightened by taking into account a random variable ζ of zero mean which accounts for an intrinsic scatter of the TF relation ($M = a.p + b - \zeta$). The method consists of two steps : the a priori choice of a statistical model, which is defined essentially on working hypotheses about the data distributions; and the derivation of parameters statistics by means of the maximum likelihood technique. This method has the advantage of providing unbiased estimates of model parameters, as long as the selection effects are taken into account by the statistical model. As standard, we assume a magnitude limited (complete) sample of uniformly distributed sources in space which shows a gaussian luminosity distribution function, although this method can easily be extended to more realistic situations. It turns clear that the presence of p -selection effects (which is not investigated here) makes this problem much more difficult, although some results require even weaker hypotheses.

We show that the “Direct Tully-Fischer” and the “Inverse Tully-Fischer” methods identify as maximum likelihood statistics related to particular models (herein, denoted ITF and DTF), whose difference limits on describing the TF diagram in a different way. At first glance, one might wonder whether such an a priori choice is justified since these models replace the one which should be prescribed by the physics of galaxies (responsible for the M - p correlation), and which is not yet known. Fortunately, it is reassuring to point out that the estimates of galaxies distances and H_0 are not model dependent, contrarily to calibration parameters a and b . Actually, these models belong to a wide class of models, and both of them can

be interpreted as a choice of a particular “orientation” for fitting the TF relation (according to usual definitions). However, the advantage of using models instead of fitting approaches is that one avoids subjective interpretations, for having clear-cut and unambiguous results. For example, we easily understand that, in order to obtain meaningful estimates, the calibration of the TF relation and the determination of H_0 , or the distances of galaxies, has to be performed within the same model, regardless of selection effects. Moreover, it turns out that the H_0 statistics are still valid when additional selection effects (or sampling rules) are present, which informs us on the robustness of these statistics. For example, in the case of the ITF model, selection effects with respect to distance modulus, or redshift, and/or M , do not perturb the estimate. On the other hand, in the case of the DTF model only additional selection effects with respect to the redshift are allowed.

The main result which ends the well known debate is that the ITF and DTF estimates show identical expectancies. Namely, the difference of estimating H_0 , resp. a distance modulus μ , by mean of ITF or DTF statistics (considered as a random variables) have vanishing mean values and a standard deviation of order of $\sigma_\zeta \gamma \sqrt{1/N_{\text{cal}} + 1/N}$, resp. $\sigma_\zeta \gamma \sqrt{1/N_{\text{cal}}}$, where N_{cal} is the size of the calibration sample, N is the size of the sample used to determine H_0 , and where the ratio $\gamma = \sigma_\zeta / \sigma_M$ informs us on the gain of accuracy when using the TF diagram. In practice, they are different only because of statistical fluctuations. It is interesting to point out that these approaches provide with us the same H_0 estimates when the calibration sample and the sample used to estimate H_0 show identical p -averages (herein called “ \mathcal{C}_p -criteria”).

Therefore, the choice between the ITF and the DTF approaches should be motivated by arguments about selection effects, accuracy and robustness of estimates. Actually, in the case of H_0 estimates, only the first criterion intervenes since the ITF and the DTF approaches show identical accuracy and robustness. With this in mind, we introduce a newly defined H_0 statistics, whose related model (herein denoted by ITF*) includes the ITF model, where no hypothesis is required on the luminosity distribution function of sources, on their spatial distribution, and it is still valid when the sample is not complete. While it is a little less accurate (by a factor of $\sqrt{1 + \gamma^2}$), its advantage is to be much more robust than the ones related to the ITF and the DTF models. Finally, simulations show that H_0 can be estimated with an accuracy of the order of 5% (1σ), which takes into account calibration and measurement errors (actually the first ones prevail on the other ones). In the case of distances, it turns out that the DTF estimate is more robust than the ITF estimate, because it does not depend on the luminosity distribution of sources. Both estimates show a correction for a bias, inadequately believed to be of Malmquist type.

A. Notations and useful formulas

The mathematical formalism is similar to the one used in Bigot & Triay (1990a). The following features are addressed throughout the text by using the symbol “Def.”.

Def.1 The *probability density* (pd) of a random variable x reads $dP(x) = f(x)dx$, where $f(x)$ represents the *pd function* (*pdf*), we have $\int dP(x) = 1$. Sometimes, it is useful to exhibit the *model parameters* involved in the statistical

model, as the mean x_0 and the standard deviation σ , by writing $f(x; x_0, \sigma)$.

(a) $g_G(x; x_0, \sigma) = (\sigma\sqrt{2\pi})^{-1} \exp - ((x - x_0)^2 / (2\sigma^2))$ is a Gaussian pdf.

(b) A normal pdf can be written $g_N(x) = g_G(x; 0, 1)$.

(c) The cumulative Normal pdf reads $\mathcal{N}(x) = \int_{-\infty}^x g_N(t) dt$.

Def.2 Let f be a pdf, and λ be a scalar value, in most of calculations, we use the following properties :

(a) $f(x + \lambda; x_0, \sigma) = f(x; x_0 - \lambda, \sigma)$;

(b) $f(\lambda x; x_0, \sigma) = \lambda^{-1} f(x; \frac{x_0}{\lambda}, \frac{\sigma}{\lambda})$;

(c) $\exp(\lambda x) g_G(x; x_0, \sigma) = \exp\left(\lambda(x_0 + \lambda \frac{\sigma^2}{2})\right) g_G(x; x_0 + \lambda \sigma^2, \sigma)$.

(d) $g_G(x; x_1, \sigma_1) g_G(x; x_2, \sigma_2) = g_G(x; x_0, \sigma_0) g_G(x_1; x_2, \sigma)$, where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$, x_0 and σ_0 are defined as follows $\sigma_0^{-2} = \sigma_1^{-2} + \sigma_2^{-2}$ and $x_0 \sigma_0^{-2} = x_1 \sigma_1^{-2} + x_2 \sigma_2^{-2}$.

Def.3 $P(h) = \int h(x) dP(x)$ denotes the expected value of the function $h(x)$.

Def.4 The pd of a sample data $\{\mathcal{G}_k\}_{k=1, N}$, which consists of N independently selected objects \mathcal{G}_k , is given by $\prod_{k=1}^N dP(\mathcal{G}_k)$.

(a) Its pdf, written in terms of *observables* (the measurable random variables), but regarded as a function of model parameters, provides us with the *likelihood function*.

(b) (*The ML method.*) The model parameters statistics are obtained by maximizing the likelihood function, or (equivalently) the natural logarithm of the *efficient part* of it, in which the terms which do not contribute to the determination of parameters are removed, herein briefly denoted by *lf*.

Def.5 We use the following usual definitions :

(a) $\langle x \rangle = \sum_{k=1}^N x_k / N$ is the average,

(b) $\text{Cov}(x, y) = \sum_{k=1}^N (x_k - \langle x \rangle)(y_k - \langle y \rangle) / (N - 1)$ is the covariance,

(c) $\Sigma(x) = \sqrt{\text{Cov}(x, x)}$ is the standard deviation,

(d) $\rho(x, y) = \text{Cov}(x, y) / (\Sigma(x)\Sigma(y))$ is the correlation coefficient.

Def.6 The problem of biases in Statistics Theory is well established : an *estimator* (or *statistic*) is biased when its expected value does not correspond to model parameter for which it has been made up. In practice, a bias is expected when the normalization factor depends on the model parameter, see Bigot & Triay (1990a, 1990b). For instance, the average of absolute magnitudes, as provided by a sample of objects brighter than a given limiting apparent magnitude, is a biased estimator of the mean intrinsic magnitude (that characterizes the population of sources). Herein, such a bias is designated as *bias of Malmquist type*, a definition which can be extended to any bias due to selection effects.

Def.7 The accuracy of an estimator is formally defined as the reciprocal of its variance (The smaller the dispersion, the greater the precision.).

B. Calculations involved in the ITF model

According to Eq. (4,8,28), the normalization factor reads

$$P_{\text{th}}(\phi_m) = \int \phi_m(M + \mu) f_M(M; M_0, \sigma_M) dM \kappa(\mu) d\mu, \quad (\text{B1})$$

which shows that it does not depend on parameters a , b and H_0 . Let us note that if we specify the functions ϕ_m , κ and f_M ,

according to hypotheses (h_1, h_2, h_3, h_4) , see Eq. (9,11,16,24), then the normalization factor is given by

$$P_{\text{th}}(\phi_m) \propto \exp \beta \left(m_{\text{lim}} - M_0 + \frac{\beta}{2} \sigma_M^2 \right). \quad (\text{B2})$$

B.1. Calibration statistics

According to Eq. (7,13,12, 19,28), the pd given in Eq. (8) reads in terms of observables as

$$dP_{\text{obs}} = \frac{\phi_m(M + \mu)}{P_{\text{th}}(\phi_m)} f_M(M; M_0, \sigma_M) dM \kappa(\mu) d\mu \times g_G(a p + b; M, \sigma_\zeta) a dp, \quad (\text{B3})$$

where the first right hand term is independent of a and b , see Eq. (B1). Therefore, the *lf* $\mathcal{L}_{\text{cal}}^{\text{ITF}}(a, b, \sigma_\zeta)$ can be written as follows

$$\mathcal{L}_{\text{cal}}^{\text{ITF}} = \ln a - \ln \sigma_\zeta - \frac{1}{N_1} \sum_{k=1}^{N_1} \frac{(a \cdot p_k + b - M_k)^2}{2\sigma_\zeta^2}. \quad (\text{B4})$$

Hence, the ML equations (obtained by equating the partial derivatives of $\mathcal{L}_{\text{cal}}^{\text{ITF}}$ with respect to a , b , and σ_ζ^2 to zero) reads

$$a \langle p(ap + b - M) \rangle_1 = \sigma_\zeta^2, \quad (\text{B5})$$

$$a \langle p \rangle_1 + b = \langle M \rangle_1, \quad (\text{B6})$$

$$\langle (ap + b - M)^2 \rangle_1 = \sigma_\zeta^2. \quad (\text{B7})$$

By expanding Eq. (B5) as $a \langle p(ap + b - M) \rangle + b \langle (ap + b - M) \rangle - \langle M(ap + b - M) \rangle = \sigma_\zeta^2$. Hence, it follows that :

– According to Eq. (B5), the first left hand term is equal to σ_ζ^2 .

– According to Eq. (B5), the second left hand term is zero.

– Thus we have $a \langle pM \rangle + b \langle M \rangle = \langle M^2 \rangle$.

– Hence, by subtracting $\langle M \rangle \times (\text{B5})$, one obtains Eq.(29).

Equation (29) follows immediately from Eq. (B5). By subtracting $a \langle p \rangle \times (\text{B5})$ to Eq. (B5), we obtain $a^2 (\langle p^2 \rangle - \langle p \rangle^2) - a (\langle pM \rangle - \langle p \rangle \langle M \rangle) = \sigma_\zeta^2$, which gives Eq. (29).

B.2. Determination of H_0

According to Eq. (7,13,12, 19,28), the pd given in Eq. (8) reads in terms of observables x , y and η , see Eq. (2,20,20), as follows

$$dP_{\text{obs}} = \frac{\phi_m(x + \eta)}{P_{\text{th}}(\phi_m)} f_M(x + \mathcal{H}; M_0, \sigma_M) \kappa(\eta - \mathcal{H}) dx d\eta \times g_G(y; \mathcal{H}, \sigma_\zeta) dy. \quad (\text{B8})$$

It is important to note that this pd reads as a product of two independent pds, and thus that

the distribution of the random variable y does not depend on x and η , whatever the form of functions f_M , κ and ϕ_m .

The integration over x and η yields

$$dP_y^{\text{ITF}^*} = g_G(y; \mathcal{H}, \sigma_\zeta) dy, \quad (\text{B9})$$

which shows that the expected value of y provides us with $P_{\text{obs}}(y) = \mathcal{H}$. Therefore, the Hubble constant can be estimated by means of the statistic given in Eq. (32). Moreover Eq. (B9) shows that the standard deviation of the y -distribution is equal to σ_ζ . Thus, the standard deviation of the statistic providing H_0 is given by Eq. (33).

If we specify the functions κ and f_M , then we can perform the ML technique. By assuming hypotheses (h_2, h_3, h_4) , see Eq. (11,16,24), the lf is given by

$$\begin{aligned} \mathcal{L}_{\text{det}}^{\text{ITF}}(H_0) = & -\frac{1}{N_2} \sum_{k=1}^{N_2} \frac{(y_k - \mathcal{H})^2}{2\sigma_\zeta^2} - \beta \mathcal{H} \\ & - \frac{1}{N_2} \sum_{k=1}^{N_2} \frac{(x_k + \mathcal{H} - M_0)^2}{2\sigma_M^2}, \end{aligned} \quad (\text{B10})$$

since the normalization factor does not depend on H_0 . Hence, the likelihood equation ($d\mathcal{L}_{\text{det}}^{\text{ITF}}/dH_0 = 0$) provides us with the statistic given in Eq. (34).

In order to estimate the accuracy of the statistic (34), we have to calculate the *pdf* of the following random variable

$$z = \frac{\sigma_M^2 y + \sigma_\zeta^2 ((M_0 - \beta\sigma_M^2) - x)}{\sigma_M^2 + \sigma_\zeta^2}. \quad (\text{B11})$$

Obvious calculations¹¹ give

$$dP_z^{\text{ITF}} = g_G(z; \mathcal{H}, \frac{\sigma_\zeta}{\sqrt{1 + \gamma^2}}) dz, \quad (\text{B12})$$

see Eq. (15). According to Eq. (29,33,B12), and since $\sigma_M \approx \Sigma_1(M)$, we obtain the accuracy given in Eq. (35).

C. Calculations involved in the DTF model

Now, according to Eq. (4,7,8, 12,19,37), it turns out that the random variables M , p and μ are correlated together¹². Hence, the normalization factor $P_{\text{th}}(\phi_m)$ becomes dependent on model parameters a and b . Thus, for proceeding with the ML technique, we have to calculate explicitly $P_{\text{th}}(\phi_m)$ and its derivatives with respect to model parameters, which forces us to presume a priori the form of functions $\phi_m(m)$, $f_p(p; p_0, \sigma_p)$ and $\kappa(\mu)$. We assume (h_1, h_2, h_3, h_4) , see Eq. (9,11,38). Hence, after little algebra¹³, it turns out that the normalization is still given by Eq. (B2), with

$$M_0 = ap_0 + b, \quad (\text{C1})$$

$$\sigma_M = \sqrt{a^2\sigma_p^2 + \sigma_\zeta^2}. \quad (\text{C2})$$

¹¹- Eq. (B8) is written in terms of variables x , η and z , accordingly to Eq. (9,11,24), - one integrates over η , and hence over x , we use (Def.2.c,d).

¹²- M and p , because of the TF diagram, - M and μ , because the selection function $\phi_m(M + \mu)$ does not split into a product of two functions, - μ and p , as a consequence of above correlations.

¹³The calculation is straightforward by means of by part integrations, successively over μ , M , and finally p , where (Def.2.c) is used twice.

C.1. Calibration statistics

According to Eq. (B2,C1, C1), which shows that $P_{\text{th}}(\phi_m)$ depends indeed on parameters a , b , p_0 , σ_p , and σ_ζ , and to Eq. (7,8,12, 19,37), the lf $\mathcal{L}_{\text{cal}}^{\text{DTF}}(a, b, \sigma_\zeta, p_0, \sigma_p)$ can be written as follows

$$\begin{aligned} \mathcal{L}_{\text{cal}}^{\text{DTF}} = & -\ln P_{\text{th}}(\phi_m) - \ln \sigma_\zeta - \frac{1}{N_1} \sum_{k=1}^{N_1} \frac{(a \cdot p_k + b - M_k)^2}{2\sigma_\zeta^2} \\ & - \ln \sigma_p - \frac{1}{N_1} \sum_{k=1}^{N_1} \frac{(p_k - p_0)^2}{2\sigma_p^2}. \end{aligned} \quad (\text{C3})$$

According to Eq. (C3), the ML equations (obtained by equating the partial derivatives of $\mathcal{L}_{\text{cal}}^{\text{DTF}}$ with respect to a , b , p_0 , σ_p^2 and σ_ζ^2 to zero) can be written

$$\langle p(ap + b - M) \rangle_1 = \beta\sigma_\zeta^2 (p_0 - \beta\sigma_p^2 a), \quad (\text{C4})$$

$$a\langle p \rangle_1 + b = \langle M \rangle_1 + \beta\sigma_\zeta^2, \quad (\text{C5})$$

$$p_0 = \langle p \rangle_1 + \beta\sigma_p^2 a, \quad (\text{C6})$$

$$\langle (p - p_0)^2 \rangle_1 = \sigma_p^2 (1 + \beta^2 \sigma_p^2 a^2), \quad (\text{C7})$$

$$\langle (ap + b - M)^2 \rangle_1 = \sigma_\zeta^2 (1 + \beta^2 \sigma_\zeta^2). \quad (\text{C8})$$

Equations (C4,C4) show that $\sigma_p = \Sigma(p)$, while the Malmquist bias intervenes in the statistic Eq. (C4). According to Eq.(C4), Eq. (C4) - $\langle p \rangle \times$ Eq. (C4) yields Eq. (39). By expanding $\langle (ap + b - M)^2 \rangle$, according to Eq.(C4,39), we obtain Eq. (39).

C.2. Determination of H_0

According to Eq. (3,7), the *pd* (8) reads in terms of observables x , y and η , see Eq. (2,20,20), as follows

$$\begin{aligned} dP_{\text{obs}} = & \frac{\phi_m(x + \eta)}{P_{\text{th}}(\phi_m)} f_p\left(\frac{x + y - b}{a}; p_0, \sigma_p\right) \\ & \kappa(\eta - \mathcal{H}) \frac{1}{a} dx d\eta \times g_G(y; \mathcal{H}, \sigma_\zeta) dy \end{aligned} \quad (\text{C9})$$

Equations (B2,C1,C1) show that the normalization factor $P_{\text{th}}(\phi_m)$ does not depend on H_0 . Thus, according to Eq. (9,11,17,38), the lf reads

$$\mathcal{L}_{\text{det}}^{\text{DTF}}(H_0) = -\frac{1}{N_2} \sum_{k=1}^{N_2} \frac{(y_k - \mathcal{H})^2}{2\sigma_\zeta^2} - \beta \mathcal{H} \quad (\text{C10})$$

Hence, the likelihood equation ($d\mathcal{L}_{\text{det}}^{\text{DTF}}/dH_0 = 0$) provides us with Eq. (42). Obvious calculations¹⁴ provide us with the *pdf* describing the distribution of the random variable y ,

$$dP_y^{\text{DTF}} = g_G(y; \mathcal{H} + \beta\sigma_\zeta^2, \sigma_\zeta) dy, \quad (\text{C11})$$

which shows that the standard deviation is given by Eq. (43).

¹⁴One integrates the *pdf* given in Eq. (C9) over η , and after over x , we use (Def.2.a,b,c), which gives Eq. (B2,C1, C1).

D. Differences on the data description

In this section, we show that the DTF model and the ITF model describe the data distribution in a different way. This statement can easily be proved by supposing the antithesis, which is that the model parameters a and b are identically defined in both models (and thus also the random variable ζ). Indeed, if a and b are the same in both models, the luminosity distribution function f_M can be calculated according to Eq. (5) but within the DTF model, as given by Eq. (37). Now, by writing the *pdf* f_p according to Eq. (6), thus within the ITF model, see Eq. (28), we obtain two integrals that we transpose for obtaining the following compatibility condition¹⁵

$$f_M(M) = \int f_M(t) g_G(t; M, \sqrt{2}\sigma_\zeta) dt, \quad (D1)$$

$$\approx f_M(M) + \sigma_\zeta^2 \partial_M^2 f(M)$$

which cannot be achieved, while such a disagreement is not so drastic as that if the luminosity distribution function varies weakly within ranges of the order of $\sqrt{2}\sigma_\zeta$.

E. Biases due to measurement errors

In order to calculate the magnitude of biases related to measurement errors, we have to calculate the normalization factor $P_{\text{th}}(P_\epsilon^{(s)}(\hat{\phi}_m))$, see Eq. (68). It turns out that one needs to specify the function $\kappa(\mu)$, and thus we assume a uniform spatial distribution of sources, i.e., (h_2) , see Eq. (11). Hence, it is clear that the integrations over the ϵ_λ give unity, excepted for the one over ϵ_m , because of selection effects, see Eq. (69). The calculation becomes evident if we use the dummy variable $\tilde{\mu} = \mu + \epsilon_m$, so we have $\kappa(\mu) = \kappa(\tilde{\mu}) \exp(-\beta\epsilon_m)$, and then by using (Def.2.b,c) we obtain Eq. (72). Hence, Eq.(68) transforms as follows

$$d\hat{P}_{\text{obs}}^{(s)} = \hat{\phi}_m \frac{dP_{\text{th}}}{P_{\text{th}}(\hat{\phi}_m)} \times \exp\left(-\frac{1}{2}(\beta\sigma_{\epsilon_m}^{(s)})^2\right) dP_\epsilon^{(s)}, \quad (E1)$$

where it becomes clear that the selection function $\hat{\phi}_m$ plays the role of a correlation function between the variables m and ϵ_m . If the measurement errors were known then one could restore the values of observables from Eq. (63-63), and then use the ML technique for obtaining genuine statistics. In such a case, according to Eq. (E1), and because one has necessarily $\hat{\phi}_m(m_k) = 1$ for all individual datum ($k = 1, N$), one understands that one still obtains identical statistics to the ones given by Eq. (32,34,42), where the errors are ignored. However, since (in practice) the measurement errors are not known, the ϵ_λ -dependent parts of these statistics are substituted by their expected value according to the *p.d.* given in Eq. (E1). Let us proceed with preliminary calculations. It is easy to show that

$$\hat{P}_{\text{obs}}^{(s)}(\epsilon_\lambda) = \begin{cases} -\beta(\sigma_{\epsilon_m}^{(s)})^2 & \text{if } \epsilon_\lambda = \epsilon_m \\ 0 & \text{otherwise} \end{cases}, \quad (E2)$$

and

$$\hat{P}_{\text{obs}}^{(s)}\left(\left(\epsilon_\lambda - \hat{P}_{\text{obs}}^{(s)}(\epsilon_\lambda)\right)^2\right) = (\sigma_{\epsilon_\lambda}^{(s)})^2. \quad (E3)$$

¹⁵The approximation is obtained by expanding the right hand term, see also (Def.2.d).

Accordingly to Eq. (7,63,63), for *Step 1*, let us define the following variables

$$\hat{M} = \hat{m} - \hat{\mu} \quad (E4)$$

$$\epsilon_M = \epsilon_m - \epsilon_\mu, \quad (E5)$$

so that the absolute magnitude reads

$$M = \hat{M} - \epsilon_M. \quad (E6)$$

Note that, because ϵ_m and ϵ_μ are independent random variables, we have

$$(\sigma_{\epsilon_M}^{(1)})^2 = (\sigma_{\epsilon_m}^{(1)})^2 + (\sigma_{\epsilon_\mu}^{(1)})^2. \quad (E7)$$

Therefore, according to Eq. (E2,E4-E7), we have

$$\langle M \rangle_1 = \langle \hat{M} \rangle_1 + \beta(\sigma_{\epsilon_m}^{(1)})^2, \quad (E8)$$

and since M and ϵ_M are independent, it follows

$$\Sigma_1^2(M) = (1 - \delta_M) \Sigma_1^2(\hat{M}), \quad (E9)$$

see Eq. (70). Similarly, it is evident to show that

$$\langle p \rangle_1 = \langle \hat{p} \rangle_1, \quad (E10)$$

$$(\Sigma_1(p))^2 = (1 - \delta_p) (\Sigma_1(\hat{p}))^2, \quad (E11)$$

see Eq. (70), and thus that

$$\rho_1^2(p, M) = \frac{\rho_1^2(\hat{p}, \hat{M})}{(1 - \delta_p)(1 - \delta_M)}. \quad (E12)$$

For convenience in writing we use the following variable

$$\epsilon_{\sigma_\zeta} = \left(\frac{\delta_p}{1 - \delta_p} + \delta_M \right), \quad (E13)$$

see Eq. (70,70). In the case of the ITF model, according to Eq. (29-29,E8-70, E13) one has

$$a^{\text{ITF}} = \hat{a}^{\text{ITF}}(1 - \delta_M), \quad (E14)$$

$$b^{\text{ITF}} = \hat{b}^{\text{ITF}} - \delta_M \hat{a}^{\text{ITF}} \langle \hat{p} \rangle_1 + \beta(\sigma_{\epsilon_m}^{(1)})^2, \quad (E15)$$

$$(\sigma_\zeta^{\text{ITF}})^2 = (1 - \delta_M) (\hat{\sigma}_\zeta^{\text{ITF}})^2 - \epsilon_{\sigma_\zeta} (1 - \delta_M) (1 - \delta_p) (\hat{a}^{\text{ITF}})^2 (\Sigma_1(\hat{p}))^2. \quad (E16)$$

Hence, according to Eq. (20), we easily obtain Eq. (73). Similarly, for the DTF model, we obtain

$$a^{\text{DTF}} = \frac{\hat{a}^{\text{DTF}}}{(1 - \delta_p)}, \quad (E17)$$

$$b^{\text{DTF}} = \hat{b}^{\text{DTF}} + \delta_M \left(\hat{a}^{\text{DTF}} \langle \hat{p} \rangle_1 - \beta(\hat{\sigma}_\zeta^{\text{DTF}})^2 \right) - \epsilon_{\sigma_\zeta} \left(\hat{a}^{\text{DTF}} \langle \hat{p} \rangle_1 - \beta(\hat{\sigma}_\zeta^{\text{DTF}})^2 + \beta(\Sigma_1(\hat{M}))^2 \right) + \beta(\sigma_{\epsilon_m}^{(1)})^2 \quad (E18)$$

$$(\sigma_\zeta^{\text{DTF}})^2 = \frac{1}{1 - \delta_M} (\hat{\sigma}_\zeta^{\text{DTF}})^2 - \epsilon_{\sigma_\zeta} (\Sigma_1(\hat{M}))^2. \quad (E19)$$

Hence, according to Eq.(20), we easily obtain Eq.(73).

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